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The torsion of a viscoelastic plate in an unsteady flow☆

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Abstract

The stability of the equilibrium position of a viscoelastic plate, subjected to torsional strain and effect of the free airflow is investigated. The unsteadiness of the flow is taken into account by introducing integral terms into the moments of the aerodynamic forces acting on the plate. In a neighbourhood of the equilibrium position, a general solution of a Volterra-type integro-differential equation with partial derivatives is constructed in the form of a Fourier series, as a function of the longitudinal coordinate of the plate with coefficients that are the power series in the small parameters introduced. The stability of the plate equilibrium in the unstrained state is analysed in the case when there are small perturbations (possibly, discontinuous) of the flow velocity. The stability under persistent perturbations of the equilibrium of the strained plate with respect to non-linear perturbing forces and perturbations of its shape at the instant of time preceding the specified initial instant is also investigated.

1. The stability of the equilibrium of a plate in the unstrained state

Consider the motion of a uniform viscoelastic plate of invariant cross-section, one end of which is clamped and the other end free, in an unsteady airflow. The plate is subject to torsional strain, where the torsion occurs around a certain horizontal axis OL fixed in the body and in space. We will use the plane sections hypothesis for the torsional strain.

We will assume that a small perturbation is superimposed on the main flow, of constant velocity V_0 directed horizontally, so that the perturbed flow velocity V (directed horizontally and perpendicular to the OL axis) is represented in the form

$$V = V_0 + \mu v_1(t) \tag{1.1}$$

where $\mu > 0$ is a small parameter and $v_1(t)$ is a piecewise continuous function, specified when $t \in R^+$ and having only a finite number of points of discontinuity in each finite segment (which is quite sufficient for practical purposes). We will draw the coordinate x axis with origin at the point O, which lies on the clamped end of the plate, through the segment OL, the length of which is equal to L. The point L is situated on its free end.

Suppose $\vartheta = \vartheta(x, t)$ is the angle of rotation around the x axis of the section of the plate $x = \text{const}$. The angle ϑ is measured from the fixed horizontal plane. The moment of the aerodynamic forces acting on the plate from the side of

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the flow with velocity V_0 will be taken in the form[1]

$$M = m_0 + m_1\vartheta(x,t) + m_2\vartheta_t(x,t) + \int_0^t I_1(t-s)\vartheta_s(x,s)ds + \int_0^t I_2(t-s)\vartheta_{ss}(x,s)ds + \\ + I_1(t)\vartheta(x,0) + I_2(t)\vartheta_t(x,0) + M'(\vartheta(x,t), \vartheta_t(x,t)) \tag{1.2}$$

The holomorphic function $M'(\vartheta, \vartheta_t)$ denotes the non-linear terms; its expansion at zero begins with third-order terms, where it is assumed that $M'(-\vartheta, -\vartheta_t) = -M'(\vartheta, \vartheta_t)$. The functions $I_i(t)$ $(i = 1, 2)$ are assumed to be continuously differentiable. The partial derivatives with respect to time and the coordinate are denoted by appropriate subscrips.

We will first consider the case when the constant moment of the aerodynamic forces m_0 is equal to zero.

Converting the integral terms of expression (1.2) by integration by parts, we will have

$$M = m_1'\vartheta(x,t) + m_2'\vartheta_t(x,t) + y_1 + y_2 + M', \quad m_i' = m_i + I_i(0), \quad i = 1, 2 \\ y_1 = \int_0^t M_1(t-s)\vartheta(x,s)ds, \quad y_2 = \int_0^t M_2(t-s)\vartheta_s(x,s)ds, \quad M_i(t) = \dot{I}_i(t) \tag{1.3}$$

A dot denotes a total derivative with respect to t.

Taking formula (1.1) into account, we will write the moment $\tilde{M}$ of the aerodynamic forces of the perturbed flow in the form

$$\tilde{M} = M(1 + \mu\psi(t)); \quad |\psi(t)| \le C_\psi; \quad C_\psi = \text{const} > 0 \tag{1.4}$$

where $\psi(t)$ is a certain function, bounded when $t \in R^+$, of the same type as $\upsilon_1(t)$ in relation (1.1).

We will represent the moment of the viscoelastic forces acting in the section of the plate with coordinate x, following Volterra,[2] confining ourselves to linear terms, as follows:

$$M_\upsilon = l\vartheta_x(x,t) - \int_0^t L_1(t-s)\vartheta_x(x,s)ds; \quad l = \text{const} > 0 \tag{1.5}$$

where the relaxation kernel $L_1(t) > 0$ (a continuous function when $t \in R^+$) and the kernels $M_i(t)$ in formula (1.3) by assumption satisfy the estimate

$$L_1(t), |M_i(t)| \le C'\exp(-\alpha t), \quad \alpha, C' = \text{const} > 0, \quad i = 1, 2 \tag{1.6}$$

Henceforth, for a certain function $f(t)$, which satisfies an equality of the type (1.6), we will use the notation $f(t) \in e_1(-\alpha)$ and say that $f(t)$ belongs to the class $e_1(-\alpha)$.

Suppose mg is the weight of a plate of unit length and r is the distance of the centre of mass of the section from the x axis. We will assume that the moment of the force of gravity about the x axis is equal to 0 when $\vartheta = 0$ and is given by the formula

$$M_g = -mgr\sin\vartheta \tag{1.7}$$

Using the Hamilton - Ostrogradskii principle for systems with distributed parameters, as previously in Refs. 3 and 4, we will set up an equation, on the basis of relations (1.3)–(1.5) and (1.7), writing it in the somewhat more general form

$$I\vartheta_{tt}(x,t) - l\vartheta_{xx}(x,t) + \int_0^t L_1(t-s)\vartheta_{xx}(x,s)ds - (m_1''\vartheta(x,t) + m_2'\vartheta_t(x,t) + y_1 + y_2) = \\ = \Phi_1(\vartheta, \vartheta_t) + \mu\psi(t)\Phi_2(\vartheta, \vartheta_t, y_1, y_2); \quad m_1'' = m_1' - mgr \tag{1.8}$$

where I is the moment of inertia of a plate of unit length about the x axis, $\Phi_1(\vartheta, \vartheta_t)$ is a non-linear holomorphic function with the properties of the function $M'(\vartheta, \vartheta_t)$, and $\Phi_2(\vartheta, \vartheta_t, y_1, y_2)$ is a holomorphic function of the variables ϑ, ϑ_t and the integrals y_1 and y_2, such that

$$\Phi_2(\vartheta, \vartheta_2, y_1, y_2) = -\Phi_2(-\vartheta, -\vartheta_t, -y_1, -y_2) \tag{1.9}$$

To investigate the stability of the equilibrium position of the plate $\vartheta(x, t) \equiv 0$, we construct the general solution of Eq. (1.8) which satisfies the boundary conditions in the neighbourhood of zero.

$$\vartheta(0, t) \equiv 0, \quad \vartheta_x(L, t) \equiv 0 \tag{1.10}$$

We will follow the general scheme for constructing the solution employed in Refs. 3 and 4.

We will represent the solution of Eq. (1.8) in accordance with conditions (1.10) in the form of a Fourier series in the system of functions

$$\sin((2k+1)\omega x); \quad \omega = \pi/(2L), \quad k = 0, 1, 2, \ldots \tag{1.11}$$

and a power series in μ

$$\vartheta(x, t) = \sum_{p=0}^{\infty} \mu^p \vartheta^{(p)}(x, t), \quad \vartheta^{(p)}(x, t) = \sum_{k=0}^{\infty}\sum_{q=0}^{\infty} T_{2k+1}^{(p, 2q+1)}(t)\sin((2k+1)\omega x) \tag{1.12}$$

Here $2q+1$ is the degree of the homogeneous form on the right-hand side of Eq. (1.8) in the variables ϑ, ϑ_t, $y_i (i=1,2)$. The value $q=0$ corresponds to the linearized equation.

We will first discuss the determination of the coefficients $T_{2k+1}^{(p,2q+1)}(t)$ in representation (1.12). For the linear unperturbed equation, i.e. when $p=0$ and $q=0$, we have

$$\begin{gathered} I\ddot{T}_{2k+1}^{(0,1)}(t) + \omega^2(2k+1)^2\left(lT_{2k+1}^{(0,1)}(t) - \int_0^t L_1(t-s)T_{2k+1}^{(0,1)}(s)ds\right) - \\ -m_1''T_{2k+1}^{(0,1)}(t) - m_2'\dot{T}_{2k+1}^{(0,1)}(t) - \int_0^t M_1(t-s)T_{2k+1}^{(0,1)}(s)ds - \int_0^t M_2(t-s)\dot{T}_{2k+1}^{(0,1)}(s)ds = 0 \end{gathered} \tag{1.13}$$

When solving Eq. (1.8) we will take the following initial conditions, which are assumed to be represented in the form of Fourier series in the system of functions (1.11)

$$\begin{aligned} \vartheta(x, 0) &= \varphi_1(x) = \sum_{k=0}^{\infty} a_{2k+1}\sin((2k+1)\omega x) \\ \vartheta_t(x, 0) &= \varphi_2(x) = \sum_{k=0}^{\infty} b_{2k+1}\sin((2k+1)\omega x) \end{aligned} \tag{1.14}$$

where a_k and b_k are specified constants, which serve as the initial data when solving the Cauchy problem for Eq. (1.13), so that

$$T_{2k+1}^{(0,1)}(0) = a_{2k+1}, \quad \dot{T}_{2k+1}^{(0,1)}(0) = b_{2k+1}$$

Denoting

$$\begin{aligned} &x_{2k+1}^{(i,j)}(t) = T_{2k+1}^{(i,j)}(t), \quad \tilde{x}_{2k+1}^{(i,j)}(t) = \dot{T}_{2k+1}^{(i,j)}(t) \\ &X_{2k+1}^{(i,j)}(t) = \text{col}(x_{2k+1}^{(i,j)}(t), \tilde{x}_{2k+1}^{(i,j)}(t)), \quad \overset{0}{x}_{2k+1} = \text{col}(a_{2k+1}, b_{2k+1}) \end{aligned} \tag{1.15}$$

we can write the general solution of Eq. (1.13) in the form

$$X_{2k+1}^{(0,1)}(t) = X_{2k+1}(t)\overset{0}{x}_{2k+1}, \quad k = 0, 1, 2, \ldots \tag{1.16}$$

where $X_{2k+1}(t-s) = (x_{ij}^{2k+1}(t-s))$ is the fundamental 2×2 matrix written in the form system of Eq. (1.13) with subscript $2k+1$ and lower limits of integration s in the integral terms, where $X_{2k+1}(0) = E_2$.

We will henceforth assume that the solution of each of Eq. (1.13) satisfies the limit

$$\|X_{2k+1}(t)\| \le C\exp(-\beta t), \quad C, \beta = \text{const} > 0, \quad k = 0, 1, 2, \ldots \tag{1.17}$$

When the functions $\varphi_1(x)$ and $\varphi_2(x)$ in (1.14) have continuous derivatives up to the third order inclusive, which possess the property of bounded variation, for the coefficients a_{2k+1} and b_{2k+1} depending on the number $2k+1$, as was noted in Refs. 3 and 4, the following relations hold

$$a_{2k+1} = O((2k+1)^{-4}), \quad b_{2k+1} = O((2k+1)^{-4}) \tag{1.18}$$

If condition (1.18) is satisfied, the Fourier series for $\vartheta^{(0)}(x, t)$ and $\vartheta_t^{(0)}(x, t)$, like those for $\vartheta_{tt}^{(0)}(x, t)$ and $\vartheta_{xx}^{(0)}(x, t)$, are convergent absolutely when $x \in [0, L]$ and $t \in R^+$, and the constructed solution of the linear unperturbed equation is not formal.

We will denote the linear operator in Eq. (1.13) by $H(T_{2k+1}^{(0,1)}(t))$. To determine the functions $T_{2k+1}^{(0,3)}(t)$, by relations (1.8) and (1.12) we have the equations

$$H(T_{2k+1}^{(0,3)}(t)) = s_{2k+1}^{(0,3)}(t), \quad k = 0, 1, \ldots \tag{1.19}$$

where $s_{2k+1}^{(0,3)}(t)$ are continuous functions, bounded in $t \in \mathrm{R}^+$, which are given by the relation

$$\sum_{k=0}^{\infty} s_{2k+1}^{(0,3)}(t)\sin((2k+1)\omega x) = \Phi_1^{(3)}(\vartheta^{(0,3)}(x,t), \vartheta_t^{(0,3)}(x,t)) \tag{1.20}$$

Here

$$\vartheta^{(0,3)}(x,t) = \sum_{k=0}^{\infty} T_{2k+1}^{(0,1)}(t)\sin((2k+1)\omega x)$$

and $\Phi_1^{(3)}(\vartheta, \vartheta_t)$ is a third-power form in the expansion of the function

$$\Phi_1(\vartheta, \vartheta_t) = \sum_{s=1}^{\infty} \Phi_1^{(2s+1)}(\vartheta, \vartheta_t)$$

with respect to the homogeneous forms of odd order. Note that the right-hand side of relation (1.20) is expanded in a Fourier series in the system of functions (1.11) by virtue of the oddness of the form $\Phi_1^{(3)}(\vartheta, \vartheta_t)$.

For zero initial conditions, the solution of the linear inhomogeneous Eq. (1.19), written in the form of a system, we will represent in the notation (1.15), assuming, in addition, that

$$S_{2k+1}^{(0,q)}(t) = \text{col}(0, I^{-1}s_{2k+1}^{(0,q)}(t))$$

as follows:

$$X_{2k+1}^{(0,3)}(t) = \int_0^t X_{2k+1}(t-s)S_{2k+1}^{(0,3)}(s)ds \tag{1.21}$$

Similarly, for arbitrary q, if all the functions $T_{2k+1}^{(0,2s+1)}(t)$ are obtained for $s = 0, 1, \ldots, q-1$, to determine $T_{2k+1}^{(0,2q+1)}(t)$ we have an equation of the form (1.19) (with the superscript 3 replaced by $2q+1$), in which the functions $s_{2k+1}^{(0,2q+1)}(t)$ are given by a relation similar to (1.20), using, on the right-hand side, the homogeneous forms $\Phi_1^{(2r+1)}(\vartheta, \vartheta_t)(r = 1, 2, \ldots, q)$ and the functions $T_{2k+1}^{(0,2s+1)}(t)\dot{T}_{2k+1}^{(0,2s+1)}(t)(s = 0, 1, \ldots, q-1)$, obtained at the previous stages of the calculation. The solution of this equation with zero initial conditions is written in the form (1.21) with the superscript 3 replaced by $2q+1$.

Hence, the solution of the unperturbed equation (1.8) in the form of a Fourier series, can be constructed at least formally.

We will estimate the function $X_{2k+1}^{(0,3)}(t)$. Taking inequality (1.17) into account and also the fact that, in view of relation (1.20) the following inequality holds

$$\left\|S_{2k+1}^{(0,3)}(t)\right\| \le C_1\exp(-3\beta t);\quad C_1 = \text{const} > 0$$

we have, by relations (1.21) and (1.17),

$$\left\|X_{2k+1}^{(0,3)}(t)\right\| = CC_1\int_0^t \exp(-\beta(t-s))\exp(-3\beta s)ds < C_3\exp(-\beta t);\quad C_3 = \text{const} > 0 \tag{1.22}$$

In the general case when $2q+1\ge 3$ we obtain, like (1.22), an estimate with the superscript 3 replaced by $2q+1$.
The Fourier series constructed for the function $\vartheta(x, t)$, which is the solution of the unperturbed Eq. (1.8), converges.
In fact, on the basis of formulae (1.12), (1.16), (1.20) and (1.21) we can write

$$\vartheta(x,t) = \sum_{k=0}^{\infty}\left[x_{11}^{(2k+1)}(t)a_{2k+1} + x_{12}^{(2k+1)}(t)b_{2k+1} + \right.$$
$$\left. + \int_0^t \sum_{q=1}^{\infty} x_{12}^{(2k+1)}(t-\tau)s_{2k+1}^{(0,2q+1)}(\tau)d\tau\right]\sin((2k+1)\omega x) \tag{1.23}$$

Similar relations also hold for $\vartheta_t(x, t)$.

For $x\in[0, L]$ and $t\ge 0$, will denote by $u_{2k+1}^{(2q+1)} > 0$ the upper limits of the modulus of terms of the series (1.12) (when $p=0$) and we will denote by $u>0$ the sum of the series with terms $u_{2k+1}^{(2q+1)}$. For $\vartheta_t(x, t)$ we will denote the corresponding quantities by $v_{2k+1}^{(2q+1)} > 0$ and $v > 0$. Then, taking into account relation (1.23), inequality (1.17), the procedure for constructing the functions $s_{2k+1}^{(0,2q+1)}(t)$ and, in particular, the formulae for converting the products of powers of trigonometric functions into a sum of sines of multiple arcs, we can put $u = v$ and obtain the following equation for determining the quantities u and v

$$u = C\left(\nu + \frac{1}{2\beta}\right)\Phi_1^*(u, u);\quad \nu = \sum_{k=0}^{\infty}(|a_{2k+1}| + |b_{2k+1}|) \tag{1.24}$$

$\Phi_1^*(u, v)$ is the majorant for the expansion of the function $\Phi_1(u, v)$ in a power series. It follows from the theory of majorizing equations, that the solution u(ν) in the form of a converging power series exists when $\nu\le\nu_0$ for certain $\nu_0>0$. Hence, the series for $\vartheta(x, t)$, $\vartheta_t(x, t)$ absolutely and uniformly converges when $\nu\le\nu_0$.

We will now consider the perturbed equation (1.8) when $u\ne 0$. To obtain the functions $T_{2k+1}^{(s,2q+1)}(t)(s = 1, 2, \ldots; k,\ q = 0, 1, 2, \ldots)$ we will have equations of the form (1.19), in which the right-hand side is a piecewise continuous absolutely integrable function $t\in R^+$, similar to the function $\psi(t)$. The solution of these equations will be understood in the Carathéodory sense[5] as a function that is absolutely continuous in each segment, for almost all t satisfying the integro-differential equation or the equivalent integral equation specified by the Cauchy formula. For each of these equations, by virtue of the assumptions made earlier regarding the functions in Eq. (1.8), the Carathéodory conditions are satisfied and, like the differential equations with discontinuous right-hand sides,[5] the main properties of the solutions (uniqueness, continuous dependence on the initial conditions and parameter, differentiability with respect to these quantities, etc.) hold. The legitimacy of taking the limit when constructing solutions in the form of series is guaranteed by the existence of corresponding absolutely converging majorizing series. The functions $\vartheta^{(s)}(x\ ,t)(s=1, 2, \ldots)$ with initial condition $\vartheta^{(s)}(x, 0)\equiv 0$ are defined, like (1.23), by the expression

$$\vartheta^{(s)}(x,t) = \sum_{k=0}^{\infty}\int_0^t\psi(\tau)\sum_{q=0}^{\infty}x_{12}^{(2k+1)}(t-\tau)P_{2k+1}^{(s,2q+1)}(\tau)d\tau\sin((2k+1)\omega x) \tag{1.25}$$

in which $P_{2k+1}^{(s,2q+1)}(t)(s = 1, 2, \ldots)$ are coefficients in the expansion of the form

$$\sum_{k=0}^{\infty} P_{2k+1}^{(s,2q+1)}(t)\sin((2k+1)\omega x)$$

for a form of power $2k+1$, occurring in the terms for μ^s of the function on the right-hand side of Eq. (1.8) after substituting the series (1.12) into this function. The coefficients $P_{2k+1}^{(s,2q+1)}(t)$ are similar to $s_{2k+1}^{(0,2q+1)}(t)$.

We will first analyse the contribution of the linear terms ($q=0$) of the functional $\psi(t)\Phi_2(\vartheta, \vartheta_t, y_1, y_2)$ in formula (1.25). Suppose γ is a certain number such that $0 < \gamma < \min(\alpha, \beta)$. For the non-integral term, linear in $\vartheta(x, t)$, according to the last two relations of (1.4) and relations (1.17) and (1.6), we have

$$\left|\int_0^t \psi(\tau)x_{12}^{(2k+1)}(t-\tau)T_{2k+1}^{(1,1)}(\tau)d\tau\right| \le$$

$$\le C_\psi C\int_0^t \exp(-\beta(t-\tau))\exp(-\gamma\tau)d\tau < \frac{C_\psi C}{\beta-\gamma}\exp(-\gamma t)$$

and also

$$\left|\int_0^t M_1(t-\tau)T_{2k+1}^{(1,1)}(\tau)d\tau\right| < \frac{C'C}{\alpha-\gamma}\exp(-\gamma t)$$

$$\left|\int_0^t M_2(t-\tau)\dot{T}_{2k+1}^{(1,1)}(\tau)d\tau\right| < \frac{C'C}{\alpha-\gamma}\exp(-\gamma t)$$

Hence, estimating the contribution to formula (1.25), for example, for one of the integral terms on the right-hand side of Eq. (1.8), we obtain the limit

$$\left|\int_0^t \psi(\tau)x_{12}^{(2k+1)}(t-s)M_1(s-\tau)T_{2k+1}^{(1,1)}(\tau)d\tau\right| < \frac{C_\psi C'C^2}{(\alpha-\gamma)(\beta-\gamma)}\exp(-\gamma t)$$

It can similarly be shown that all the other terms of representation (1.25) tend exponentially to zero as $t \to +\infty$, having a general exponentially decreasing factor $\exp(-\gamma t)$, and this property occurs for all $\vartheta^{(p)}(x, t)$ when $p \ge 0$. For the majorizing series $u \gg \vartheta(x, t)$, $v \gg \vartheta_t(x, t)$ we obtain two equations which, assuming, as previously, that $u = v$, can be reduced to a single equation

$$u = C\nu + C\left\{\frac{\Phi_1^{(3)*}(u,u)}{2\beta} + \mu\left[\left(\left(|m_1''| + |m_2'| + \frac{CC'}{\alpha-\gamma}\right)\frac{u}{\beta-\gamma} + \frac{\Phi_2^{(3)*}(u,u)}{2\gamma}\right]\right\} \tag{1.26}$$

When setting up Eq. (1.26), in order to simplify the expression we assumed that the linear terms of the functional $\Phi_2(\vartheta, \vartheta_t, y_1, y_2)$ are given by formula (1.3), while terms of the third and higher degrees do not contain integral terms and are given by the function $\Phi_2^{(3)}(\vartheta, \vartheta_t)$, where $\Phi_2^{(3)*}(u, v) \gg \Phi_2^{(3)}(u, v)$. From Eq. (1.26) we can obtain the condition which the parameter μ must satisfy in order that the solution of Eq. (1.8) should take the form of a power series in μ. For the case when $\Phi_2^{(3)} = \Phi_2^{(3)}(\vartheta, \vartheta_t, y_1, y_2)(i = 1, 2)$, the majorizing equation is constructed using a well-known scheme (see, for example, Ref. 3, Section 1).

It follows from the properties of the series (1.12) that $\vartheta(x, t) \to 0$ exponentially as $t \to +\infty$ uniformly in $x \in [0, L]$ and $\mu \in [0, \mu^0]$ for certain $\mu^0 > 0$.

The Fourier series obtained for $\vartheta(x, t)$ satisfies Eq. (1.8) at least formally. In order that the function $\vartheta(x, t)$ should actually be a solution of Eq. (1.8), it is sufficient to show that the Fourier series for $\vartheta_{tt}(x, t)$ and $\vartheta_{xx}(x, t)$ converge absolutely when $x \in [0, L]$ and $t \in [0, +\infty]$.

To determine the function $\vartheta_t(x, t)$ we have a formula similar to the formula for $\vartheta^{(s)}(x, t)$ (1.25). The absolutely converging series for $\vartheta_t(x, t)$ is a function differentiable with respect to t. Differentiating the formula for $\vartheta_t(x, t)$ we obtain the expression

$$\vartheta_{tt}(x,t) = \sum_{k=0}^{\infty}\left[\dot{x}_{21}^{(2k+1)}(t)a_k + \dot{x}_{22}^{(2k+1)}(t)b_k + \sum_{s=0}^{\infty}\mu^s\sum_{q=1}^{\infty}\left(p_{2k+1}^{(s,2q+1)}(t) + \int_0^t x_{22}^{(2k+1)}(t-\tau)p_{2k+1}^{(s,2q+1)}(\tau)d\tau\right)\right]\sin((2k+1)\omega x) \quad (1.27)$$

where $p_{2k+1}^{(s,2q+1)}(t)$ are piecewise continuous functions of the form $\psi(t)$, which are constructed in the same way as the functions $p_{2k+1}^{(s,2q+1)}(t)$ in representation (1.25). The elements of the matrix $\dot{X}_{2k+1}(t) = (\dot{x}_{ij}^{(2k+1)}(t))$ occur in formula (1.27).

It can be shown that $\dot{x}_{ij}^{(2k+1)}(t) \in e_1(-\gamma)$. Hence, to construct the Fourier series and the majorizing series of the function $\vartheta_{tt}(x, t)$ the above procedure is correct, and, according to representation (1.27), the majorant $v' \gg \vartheta_{tt}(x, t)$ is expressed in explicit form in terms of the majorants u and v and the parameters μ and ν. Series (1.27) converges absolutely and uniformly in $x \in [0, L]$ and $t \in [0,+\infty]$ and is a function continuous in x for each fixed $t \in R^+$ and piecewise continuous in t for each $x \in [0, L]$.

We will consider Eq. (1.8) as an integral equation for determining the derivative of $\vartheta_{xx}(x, t)$ and we will write it in the form

$$l\vartheta_{xx}(x,t) - \int_0^t L_1(t-s)\vartheta_{xx}(x,s)ds = \Phi(\mu, \vartheta, \vartheta_t, y_1, y_2) \quad (1.28)$$

where, after substituting the solution $\vartheta(x, t)$, $\vartheta_t(x, t)$ in the form of the Fourier series obtained, the right-hand side of $\Phi(\mu, \vartheta, \vartheta_t, y_1, y_2)$ becomes a known bounded (piecewise continuous) function of $t \in R^+$.

If no restraints are imposed on the value of the kernel $L_1(t)$, as was done earlier in Refs. 3 and 4, it can be shown, basing ourselves on the well-known results obtained in Refs. 6 and 7, that the Volterra type integral equation (1.28) may be solved for $\vartheta_{xx}(x, t)$ for any bounded continuous functions $L_1(t)$ and functions $\Phi(\mu, \vartheta(x, t), \vartheta_t(x, t), y_1(x, t), y_2(x, t))$, bounded and absolutely integrable with respect to t for $t \in [0, T]$, where T is a finite quantity as large as desired. Consequently, the series $\vartheta_{xx}(x, t)$ will be absolutely convergent when $x \in [0, L]$ and $t \in [0, T]$.

Hence, we can formulate the following assertion, which generalizes the corresponding result obtained in Ref. 4.

Theorem 1. *Suppose conditions (1.6), (1.9) and (1.17) are satisfied for Eq. (1.8) with the above-defined functions* $L_1(t)$, $M_i(t) (i = 1, 2)$, $\Phi_1(\vartheta, \vartheta_t)$, $\Phi_2(\mu, \vartheta, \vartheta_t, y_1, y_2)$, *and the perturbation of the flow is specified by formula (1.4) with a piecewise continuous function* $\psi(t)$ *when* $t \geq 0$. *Suppose the initial values* $\varphi_1(x)$ *and* $\varphi_2(x)$ *(1.14) possess third-order continuous derivatives of bounded variation.*

Then

1) the equilibrium position $\vartheta(x, t) \equiv 0$ of the viscoelastic plate is asymptotically stable under persistent perturbations;
2) the general solution of Eq. (1.8) in the neighbourhood of zero is represented by the Fourier series (1.12), which converges absolutely and uniformly when $x \in [0, L]$, $t \in [0,+\infty)$ for μ, a_k and b_k such that

$$\nu = \sum_{k=0}^{\infty}|a_{2k+1}| + |b_{2k+1}| \leq \nu^0, \quad \mu \leq \mu^0$$

where μ^0 and ν^0 are defined by the majorizing Eq. (1.26).

Note that the quantity ν is used as a measure of the initial perturbations in the definition of the stability, while the measure of the perturbation at the instant t is specified by the quantity $|\vartheta(x, t)| + |\vartheta_t(x, t)|$ for each $x \in [0, L]$.

2. The stability of the equilibrium of the strained plate

In the previous section we investigated the stability of the equilibrium position $\vartheta(x, t) \equiv 0$ for Eq. (1.8), in which we assumed that the constant $m_0 = 0$. We will consider the case when the constant moment of the aerodynamic forces m_0 is positive and that the plate in the equilibrium position is in a strained state.

We will investigate the motion of the plate taking into account aftereffect related to all the preceding instants of time. Assuming the lower limits of integration in formulae (1.2) and (1.5) to be equal to $[-\infty]$ and assuming that the flow is not perturbed, we arrive at the equation

$$I\vartheta_{tt}(x,t) - l\vartheta_{xx}(x,t) + \int_{-\infty}^{t} L_1(t-s)\vartheta_{xx}(x,s)ds - \left(m_0 + m_1''\vartheta(x,t) + m_2'\vartheta_t(x,t) + \int_{-\infty}^{t} M_1(t-s)\vartheta(x,s)ds + \int_{-\infty}^{t} M_2(t-s)\vartheta_s(x,s)ds\right) = \mu F(\vartheta, \vartheta_t) \tag{2.1}$$

where $\mu > 0$ is a small parameter and $F(\vartheta, \vartheta_t)$ is an odd holomorphic function, so that

$$F(\vartheta, \vartheta_t) = \sum_{i, j, k = 1 (i + j = 2k+1)}^{\infty} f_{ij}\vartheta^i\vartheta_t^j, \quad f_{ij} = \text{const}$$

and the remaining notation of Section 1 is retained. We will consider the quantity $\mu F(\vartheta, \vartheta_t)$ as a persistent perturbation.

The linearized Eq. (2.1) has the stationary solution $\vartheta(x, t) \equiv \vartheta_0(x)$, which satisfies the equation

$$L_0\ddot{\vartheta}_0(x) + M_0\vartheta_0(x) + m_0 = 0; \quad L_0 = l - \int_0^{\infty} L_1(s)ds, \quad M_0 = m_1 - mgr \tag{2.2}$$

and the boundary condition on the free end $\dot{\vartheta}_0(L) = 0$, namely,

$$\frac{M_0}{m_0}\vartheta_0(x) = \frac{\cos(\lambda_0(x \pm L))}{\cos(\lambda_0 L)} - 1; \quad \lambda_0 = \sqrt{\frac{M_0}{L_0}} \tag{2.3}$$

where the minus (plus) sign is taken if the angle $\lambda = \lambda_0 L$ in the first (second) quadrant. Similar expressions are obtained for the third and fourth quadrants. Functions (2.3) satisfy the boundary conditions and the condition $\dot{\vartheta}_0(0) > 0$.

We will assume that $L_0 > 0$, $M_0 > 0$ in Eq. (2.2).

We will further consider the case when the minus sign is taken in Eq. (2.3). The first part of this equation is first determined in the segment $[0, L]$. We will supplement it as an odd function in the segment $[-L, L]$ and then in the segment $[-L, 3L]$ as an even function with respect to the straight line $x = L$. We will supplement the function obtained in this way in the segment $[-L, 3L]$ as a periodic function with period $4L$ over the whole numerical x axis. This function will everywhere have continuous first and second derivatives, apart from the points $x = 4Lk$ ($k = 0, 1, 2\ldots$) for the second derivative where the latter has discontinuities of the first kind when $m_0 \neq 0$. In view of this fact, the function $\vartheta_0(x)$ is represented by a Fourier series in eigenvalues of the linearized unperturbed problem

$$\vartheta_0(x) = \sum_{k=0}^{\infty} C_{2k+1}^0 \sin((2k+1)\omega x) \tag{2.4}$$

where we have the estimate $C_n^0 = O(n^{-2})$ for the coefficients of this series,[8] and series (2.4) converges absolutely and uniformly for all $x \in [0, L]$ and satisfies Eq. (2.2) at least formally.

We will investigate Eq. (2.1) in the neighbourhood of the equilibrium position $\vartheta = \vartheta_0(x)$. Assuming

$$\vartheta(x, t) = \vartheta_0(x) + \theta(x, t) \tag{2.5}$$

we change to an equation of the perturbed motion in the variable $\theta(x, t)$.

Note that the function $\mu F(\vartheta, \vartheta_t)$ on the right-hand side of this equation, in view of equality (2.5), is represented by a power series, converging in a certain neighbourhood of zero, made up of homogeneous forms of odd power of the variables $\vartheta_0(x)$, $\theta(x, t)$, $\theta_t(x, t)$. We will split each of the integrals on the left-hand side of the equation into two parts: from $-\infty$ to 0 and from 0 to t. We will assume that the functions $\theta(x, t)$, $\theta_t(x, t)$ are specified for $t \in (-\infty, 0)$ and represented by the Fourier series

$$\theta(x, t) = \eta(x, t) = \mu_0 \sum_{k=0}^{\infty} \eta_{2k+1}(t)\sin((2k+1)\omega x), \quad \mu_0 = \text{const} \ll 1 \tag{2.6}$$

and by a similar series for $\theta_t(x, t) = \eta'(x, t)$ with Fourier coefficients $\eta'_{2k+1}(t)$.

We will assume that the functions $\eta_{2k+1}(t)$, $\eta'_{2k+1}(t)$ in representation (2.6) are continuous when $-\infty < t < 0$ and the following limits hold for them

$$|\eta_{2k+1}(t)|, |\eta_{2k+1}(t)| \le \frac{C''}{(2k+1)^4}, \quad C'' = \text{const} > 0, \quad k = 1, 2, \dots \tag{2.7}$$

which are satisfied if $\eta(x, t)$, $\eta'(x, t)$ have continuous third derivatives with respect to x with bounded variation when $x \in [0, L]$.

We will further assume that the functions $\eta(x, t)$, $\eta'(x, t)$ $(-\infty < t < 0)$ are persistent perturbations. As a result of this procedure additional small perturbations appear in the equation of perturbed motion, and this equation takes the form

$$U(\theta(x, t)) = \mu_0 F_0(x, t) + \mu F(\vartheta(x, t), \vartheta_t(x, t)) \tag{2.8}$$

where we have used the operator $U(\vartheta(x, t))$ on the left-hand side of Eq. (1.8) and we have introduced the following notation for the known continuous functions when $t \in \mathrm{R}^+$

$$F_0(x, t) = \sum_{i=1}^{3} \sum_{n=0}^{\infty} \xi_{2n+1}^{(i)}(t)\sin((2n+1)\omega x)$$

$$\xi_{2n+1}^{(1)}(t) = -(2n+1)^2\omega^2 l \int_{-\infty}^{0} L_1(t-s)\eta_{2n+1}(s)ds \tag{2.9}$$

$$\xi_{2n+1}^{(2)}(t) = \int_{-\infty}^{0} M_1(t-s)\eta_{2n+1}(s)ds, \quad \xi_{2n+1}^{(3)}(t) = \int_{-\infty}^{0} M_2(t-s)\eta'_{2n+1}(s)ds$$

The solution of Eqs. (2.8), (2.9) will be sought in the form

$$\theta(x, t) = \sum_{n, m=0}^{\infty} \mu^n \mu_0^n \theta^{(n, m)}(x, t) \tag{2.10}$$

assuming that

$$\theta^{(n, m)}(x, t) = \sum_{k=0}^{\infty} T_{2k+1}^{(n, m)}(t)\sin((2k+1)\omega x) \tag{2.11}$$

To determine the function $\theta^{(0, 0)}(x, t)$ we have Eq. (2.8) with zero right-hand side and initial conditions

$$\theta^{(0, 0)}(x, 0) = \varphi_1(x), \quad \theta_t^{(0, 0)}(x, 0) = \varphi_2(x) \tag{2.12}$$

which we will assume can be represented in the form (1.14). The functions $\varphi_1(x)$, $\varphi_2(x)$ by assumption have a continuous first derivative of bounded variation. Note that, generally speaking, $\varphi_1(x) \neq \eta(x, 0)$ and $\varphi_2(x) \neq \eta'(x, 0)$.

The functions $T_{2k+1}^{(0,0)}(t)$, according to relations (2.8) and (2.11), must satisfy the following equations (in the notation of (1.19))

$$H(T_{2k+1}^{(0, 0)}(t)) = 0, \quad k = 0, 1, 2, \dots \tag{2.13}$$

with initial conditions

$$T_{2k+1}^{(0,0)}(0) = a_{2k+1}, \quad \dot{T}_{2k+1}^{(0,0)}(0) = b_{2k+1}$$

As previously, we put

$$x_{2k+1}^{(n,m)}(t) = T_{2k+1}^{(n,m)}(t), \quad \tilde{x}_{2k+1}^{(n,m)}(t) = \dot{T}_{2k+1}^{(n,m)}(t)$$

and denote the fundamental 2×2 matrix of the system, corresponding to each of Eq. (2.13), by $X_{2k+1}(t-s)$, in which the lower limits of integration are replaced by s.

As previously, we will assume that inequalities (1.17) are satisfied. Assuming

$$X_{2k+1}^{(n,m)}(t) = \operatorname{col}(x_{2k+1}^{(n,m)}(t), \tilde{x}_{2k+1}^{(n,m)}(t)), \quad \Xi_j^{(i)} = \operatorname{col}(0, I^{-1}\xi_j^{(i)})$$

we will represent the general solution of Eq. (2.13) in a form similar to (1.16), and we will write the solution $X_{2k+1}^{(0,1)}(t)$ with zero initial conditions on the basis of relations (2.8)–(2.11) in the form

$$X_{2k+1}^{(0,1)}(t) = \int_0^t X_{2k+1}(t-s)\sum_{i=1}^3 \Xi_{2k+1}^{(i)}(s)ds \tag{2.14}$$

We will analyse solution (2.14), for which we estimate the functions $\xi_j^{(i)}(t)$ (2.9). Taking inequalities (1.17) and (2.7) into account we obtain

$$\left|\xi_{2k+1}^{(1)}(t)\right| \le l\omega^2\frac{C'C''}{(2k+1)^2}\int_{-\infty}^0 \exp(-\alpha(t-s))ds = \frac{l\omega^2 C'C''}{\alpha(2k+1)^2}\exp(-\alpha t) \tag{2.15}$$

i.e. $\xi_{2k+1}^{(1)}(t) \in e_1(-\alpha)$. Limits of a similar kind also hold for the other functions of (2.9), so that we have $\xi_{2k+1}^{(i)}(t) \in e_1(-\alpha)(i = 1, 2, 3)$. Hence, in view of relations (2.14) and (2.15) we obtain

$$X_{2k+1}^{(0,1)}(t) \in e_1(-\gamma), \quad 0 < \gamma < \min(\alpha, \beta)$$

We will determine the functions $\theta^{(1,0)}(x, t)$. In the notation of (2.8) we have, as previously, the equation

$$U(\theta^{(1,0)}(x,t)) = F(\vartheta_0(x,t) + \theta^{(0,0)}(x,t), \theta_t^{(0,0)}(x,t))$$

in which the right-hand side is expanded in a Fourier series in the system of functions (1.11) with coefficients $s_{2k+1}^{(1,0)}(t)$. The functions $X_{2k+1}^{(1,0)}(t)$ are given by formulae of the type (2.14) with the sum in the integrand replaced by $s_{2k+1}^{(1,0)}(s)$. In a similar way we find all the functions $X_{2k+1}^{(n,m)}(t)$ $(k, n, m = 0, 1, 2, \ldots)$, and, consequently, series (2.10) and (2.11) can be constructed. Note here that

$$\left\|X_{2k+1}^{(n,m)}(t)\right\| \le \text{const}; \quad X_{2k+1}^{(n,m)}(t) \in e_1(-y), \quad \forall m > 0, \quad t \in R^+$$

If $u \gg \theta(x, t)$, $v \gg \theta_t(x, t)$ we will have for the majorants u and v the equations

$$u = v = C\left(\nu + \frac{\mu_0}{\beta-\gamma}C_\xi + \frac{\mu}{\beta}\right)F^*(\nu_0 + u, u) \tag{2.16}$$

in which $F^*(u, v)$ is the majorant of the function $F(u, v)$, the quantity ν is defined by the last equality of (1.24) and $\nu_0 = |C_0^0| + |C_1^0| + \ldots$, $C_\xi > 0$ is a constant, such that

$$\sum_{i=1}^3\sum_{k=0}^\infty (\xi_{2k+1}^{(i)})^2 \le C_\xi$$

A positive solution of Eq. (2.16) exists in the neighbourhood of zero (and is unique) in the form of a power series in ν, ν_0, μ, μ_0, and, consequently, series (2.10), (2.13) and a similar series for $\vartheta_t(x, t)$ converge absolutely and uniformly. Hence, we have the following result.

Theorem 2. *Suppose conditions (1.17) are satisfied for Eq. (2.1) with continuous kernels $L_1(t)$, $M_i(t)$ $(i = 1, 2)$ for $t \in R^+$, which satisfy inequality (1.6), and also for the odd holomorphic non-linear function $\mu F(\vartheta, \dot{\vartheta})$. Suppose, when $-\infty < t < 0$, the values of the functions*

$$\vartheta(x, t) = \vartheta_0(x) + \eta(x, t), \quad \vartheta_t(x, t) = \eta'(x, t)$$

where $\vartheta_0(x)$ is the stationary solution (2.4) of Eq. (2.2), are assumed to be specified, and the functions $\eta(x, t)$, $\eta'(x, t)$ are continuous in t and possess continuous third derivatives with respect to x with bounded variation. Suppose when $t = 0$ the initial values of $\varphi_1(x)$, $\varphi_2(x)$ (2.12) are given, and these functions have a continuous first derivative of bounded variation.

Then

1) the general solution of Eq. (2.1) in a certain neighbourhood of the stationary state $\vartheta(x, t) \equiv \vartheta_0(x)$ can be represented in the form of series (2.5), (2.10) and (2.11); the series for $\vartheta(x, t)$, $\vartheta_t(x, t)$ converge absolutely and uniformly when

$$t \in R^+, \quad x \in [0, L], \quad \nu \le \nu^0, \quad \nu_0 \le \nu_0^0, \quad \mu \le \mu^0, \quad \mu_0 \le \mu_0^0$$

where the positive constants ν^0, ν_0^0, μ^0 and μ_0^0 are defined on the basis of the majorizing Eq. (2.16);
2) the stationary state is stable under persistent perturbations if the functions $\mu F(\vartheta, \vartheta_t)$, $\mu\eta(x, t)$, $\mu\eta'(x, t)$ are assumed to be small perturbations.

If we put $\eta(x, t) \equiv \eta'(x, t) \equiv 0$, then Theorem 2 is transformed into the corresponding result for Eq. (2.1), in which the aftereffect begins from the initial instant $t = 0$.

It follows from the structure of the general solution that when $t \to +\infty$ the plate tends to a stationary position with torsion angle $\vartheta^0(x)$, given (apart from a term that is linear in μ) by the formula

$$\vartheta^0(x) = \vartheta_0(x) + \mu \int_0^\infty \sum_{n=0}^\infty X_{2n+1}(s) ds \alpha_{2n+1} \sin((2n+1)\omega x)$$

where α_{2n+1} are the Fourier coefficients of the function $F(\vartheta_0(x), 0)$ into which series (2.4) is substituted.

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